Some Topics in Algebraic Combinatorics (Young Tableaux)

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Outline

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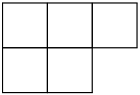
1. Examples of Combinatorics Problems
2. Definition of Posets
3. Young tableaux



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- The number of equivalence class of colorings which don't equal their 180° rotation is

$$\frac{1}{2}(n^5 - n^3)$$

Burnside Lemma:

- Let Y be a finite set and G a subgroup of a symmetric group. For each $\pi \in G$, let

$$\mathrm{Fix}(\pi) = \{y \in Y : \pi(y) = y\},$$

so $\#\mathrm{Fix}(\pi)$ is the number of cycles of length one in the permutation π. Let Y/G be the set of orbits of G. Then

$$|Y/G| = \frac{1}{\#G} \sum_{\pi \in G} \#\mathrm{Fix}(\pi)$$

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- The total number of equivalence class is $\frac{1}{2}(n^3 + n^5)$
- The total number of rectangle colorings which equal their 180° is n^3
- The number of equivalence class of colorings which don't equal their 180° rotation is $\frac{1}{2}n^5 - \frac{1}{2}n^3$

Burnside Lemma: Example

Let X be the 2×2 chessboard and let it be labeled as

1	2
3	4

$C = \{r, b, y\}$. A typical coloring can be

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y	r

How many ways can we color a 2×2 chessboard with two red squares, one blue square and one yellow square?

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How many ways can we color a 2×2 chessboard with two red squares, one blue square and one yellow square? 12

we denote Y is the set of all 12 colorings.

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- G_5 is the group of all rotations and reflections.
- G_6 is the symmetric group of all 24 permutations of Y.

For each of these groups, how many inequivalence classes do we get? For this we can use Burnside Theorem.

Burnside Lemma: Example(G_1)

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- There are 12 colorings in all with two red squares, one blue square, and one yellow square, and all are inequivalent under the trivial group.

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- There are 6 inequivalent colorings.

r	r
b	y

r	b
r	y

r	y
r	b

b	y
r	r

r	b
y	r

r	y
b	r

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$$|Y/G_3| = \frac{1}{2}(12 + 2) = 7$$

- There are 7 inequivalent colorings.

r	r	r	r	b	y	y	b	r	b	b	r	y	r
b	y	y	b	r	r	r	r	y	r	r	y	r	b

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- $G_4 = \{(1), (1243), (14)(23), (1342)\}$ is the group of all rotations.
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- There are 3 inequivalent colorings.

r	r
y	b

r	r
b	y

r	b
y	r

Burnside Lemma: Example(G_5)

- G_5 is the group of all rotations and reflections. Therefore,

$$G_5 = \{(1), (1243), (14)(23), (1342), (12)(34), (13)(24), (23), (14)\}$$

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$$\#\text{Fix}(1) = 12, \#\text{Fix}(14) = 2, \#\text{Fix}(23) = 2$$

- $$|Y/G_5| = \frac{1}{8}(12 + 2 + 2) = 2$$

- There are 2 inequivalent colorings.

r	r
b	y

r	b
y	r

Burnside Lemma: Example(G_6)

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Burnside Lemma: Example(G_6)

- G_6 is the symmetric group of all 24 permutations of X.
- There is only one equivalence class.

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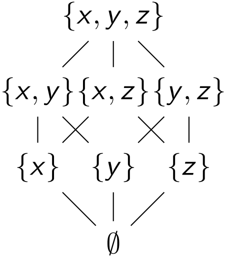

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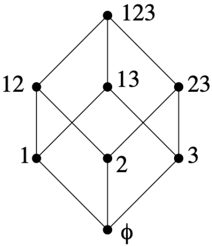
- Assume that $S = \{1, 2, 3\}$ is a 3-element set and P contains all its subsets.
- P in this case called a finite boolean algebra of rank 3 and is denoted by $B_{\{1,2,3\}}$.
- If $x, y \in P$,then define $x \leq y$ in P if $x \subseteq y$ as sets.

Hasse Diagrams

- Hasse diagram can be a simple way to represent small posets.



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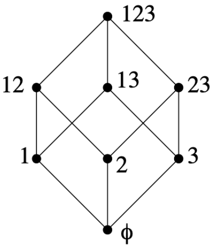

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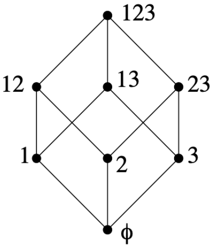

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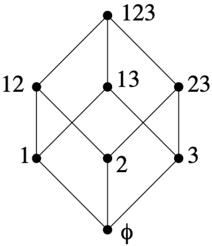

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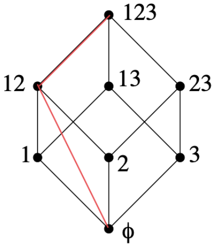
- By the transitivity property (P3), all the relations of a finite poset are determined by the cover relations, so the Hasse diagram determines P.

Chain



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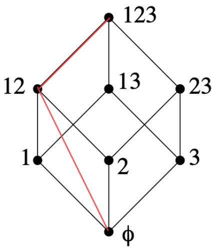

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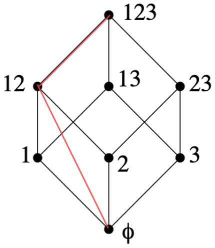

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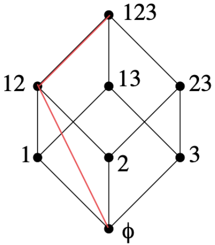

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- $\{\emptyset, \{1\}, \{1,2\}, \{1,2,3\}\}$ has length 3.
  $\{\emptyset, \{1\}, \{1,2\}\}$ has length 2.

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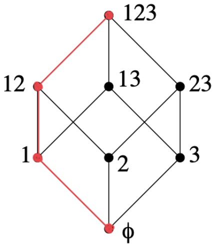

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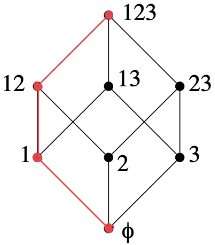

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- E.g. $\{\emptyset, \{1\}, \{1,2\}, \{1,2,3\}\}$, $\{\emptyset, \{1\}, \{1,3\}, \{1,2,3\}\}$, $\{\emptyset, \{2\}, \{1,2\}, \{1,2,3\}\}$ are all maximal and many more.

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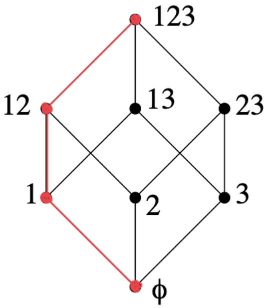

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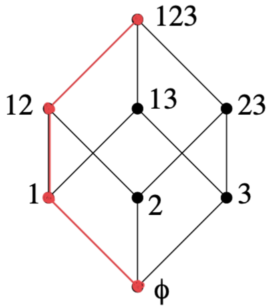

- A finite poset is *graded* of rank n if every maximal chain has length n.
- $P = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$ is finite and has rank 3

Partition

- A partition λ of an integer $n \geq 0$ is a sequence $\lambda = (\lambda_1, \lambda_2, \cdots)$ of integers $\lambda_i \geq 0$ satisfying $\lambda_1 \geq \lambda_2 \geq \cdots$ and $\sum_{i \geq 1} \lambda_i = n$.

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- All but finitely many λ_i are equal to 0.
- Only the $\lambda_i > 0$ will be called a part of λ.

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- A partition λ of an integer $n \geq 0$ is a sequence $\lambda = (\lambda_1, \lambda_2, \cdots)$ of integers $\lambda_i \geq 0$ satisfying $\lambda_1 \geq \lambda_2 \geq \cdots$ and $\sum_{i \geq 1} \lambda_i = n$.
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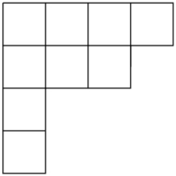
- The seven partitions of 5 are (5), $(4, 1)$, $(3, 2)$, $(3, 1, 1)$, $(2, 2, 1)$, $(2, 1, 1, 1)$, and $(1, 1, 1, 1, 1)$.

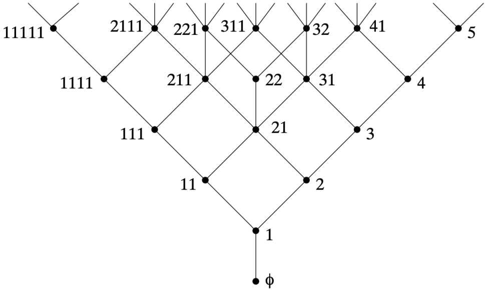
Young Diagram

- The Young Diagram (Diagram) of a partiton λ is a left-justified array of squares, with λ_i squares in the ith row.



- The example above is the Young diagram of $(4, 3, 1, 1)$.

Hasse Walk



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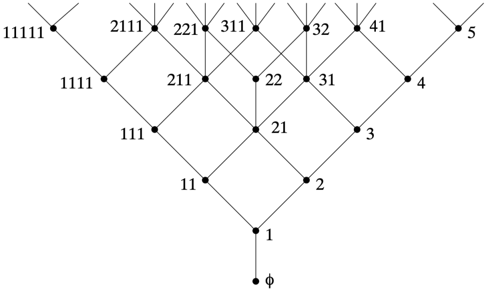

- Hasse diagram of Y has no loops or multiple edges, a walk of length n is specified by a sequence $\lambda_0, \lambda_1, \cdots, \lambda_n$ of vertices of Y .
- A walk in the Hasse diagram of a poset is a *Hasse walk* (or just walk for this section.)

Young tableaux

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- If the walk W has steps of types $A_1, A_2, \cdots, A_{n-1}, A_n$, respectively, where each A_i is either U or D, then we say that W is of type

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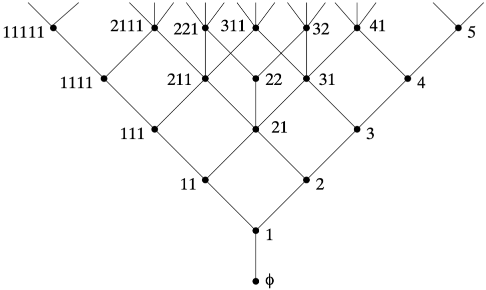
- The reason that the type of a walk is written in the opposite order to that of the walk is because we regard U and D as linear transformations.

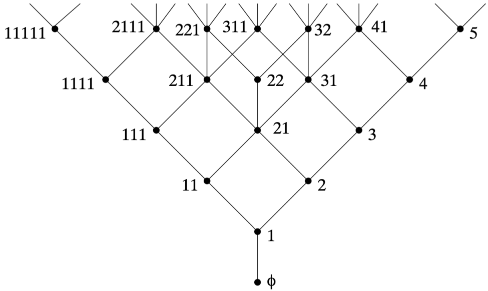
Example



- The walk $\emptyset, 1, 2, 1, 11, 111, 211, 21, 22, 21, 31, 41$ is of type $UUDDUUUUDUU = U^2D^2U^4DU^2$.

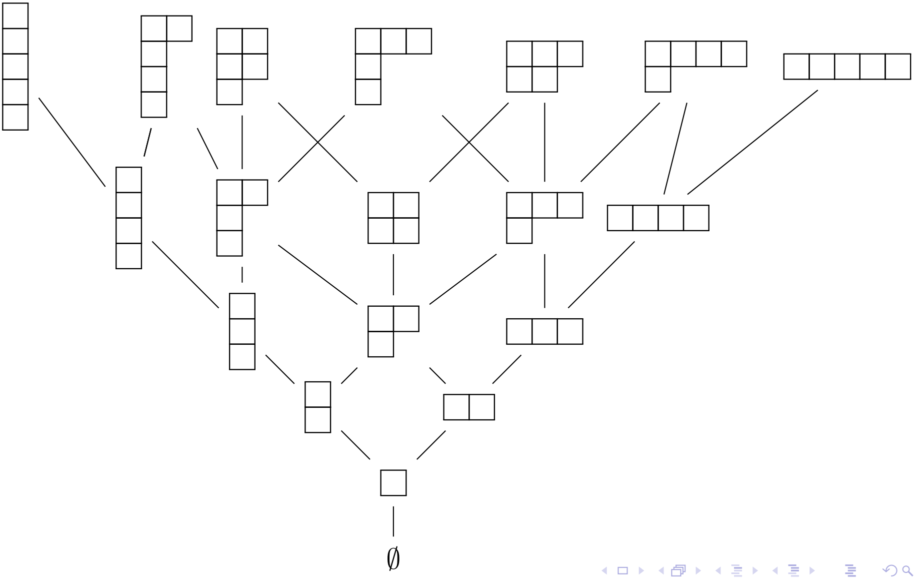
The Walks of Type U^n which begin at $\emptyset$

- The Walks of Type U^n which begin at $\emptyset$ are just saturated chain $\emptyset = \lambda^0 \lessdot \lambda^1 \lessdot \cdots \lessdot \lambda^n$.

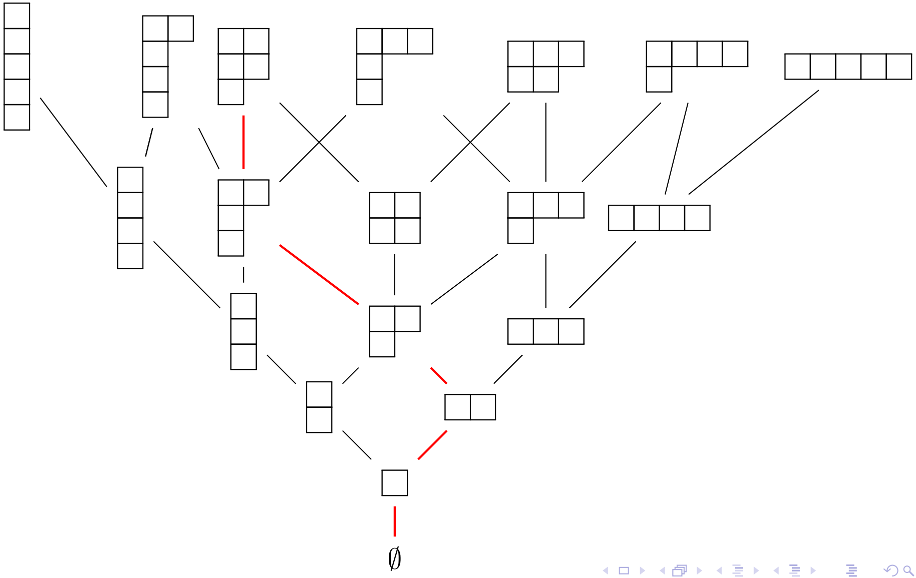


- Goal: Count the number of walks of type U^n.

Young Lattice



An example of U^5



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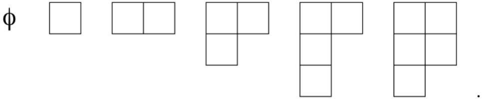

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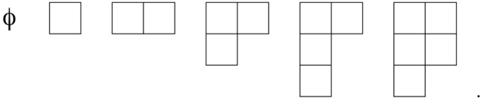

- We can visualize it on the Hasse diagram.

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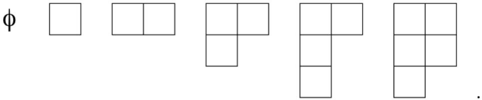

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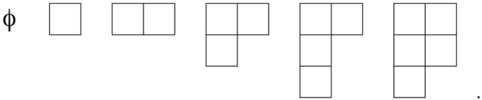
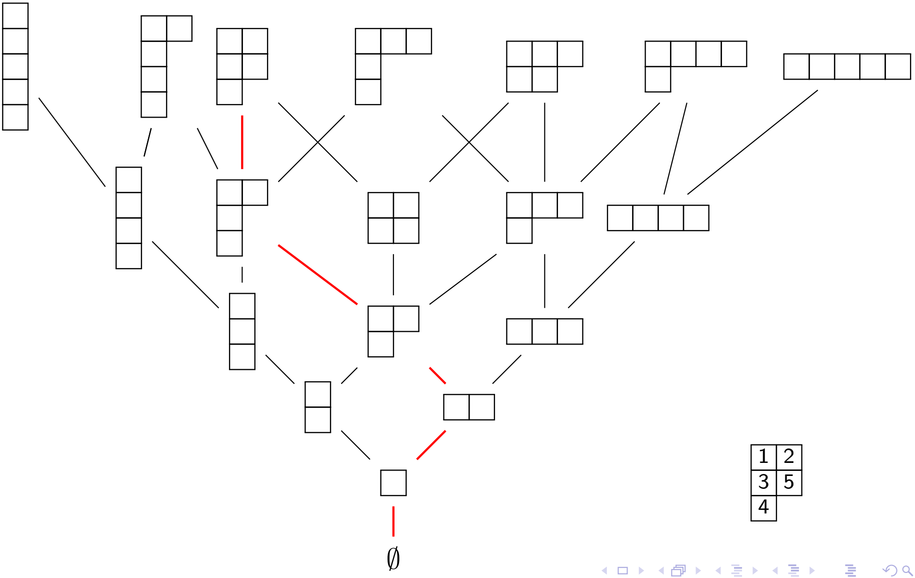


- We can specify the walk by taking the final diagram and inserting an i into square s if s was added at the ith step.
- Thus the above walk is encoded by the "tableau"

1	2
3	5
4	

An example of U^5



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- The standard Young Tableaux consists of the Young diagram D of some partition λ of an integer n, together with number number $1, 2, \ldots, n$ inserted into the squares of D.

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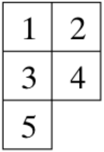

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- Define f^λ the number of SYT of shape λ.

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2	4
5	

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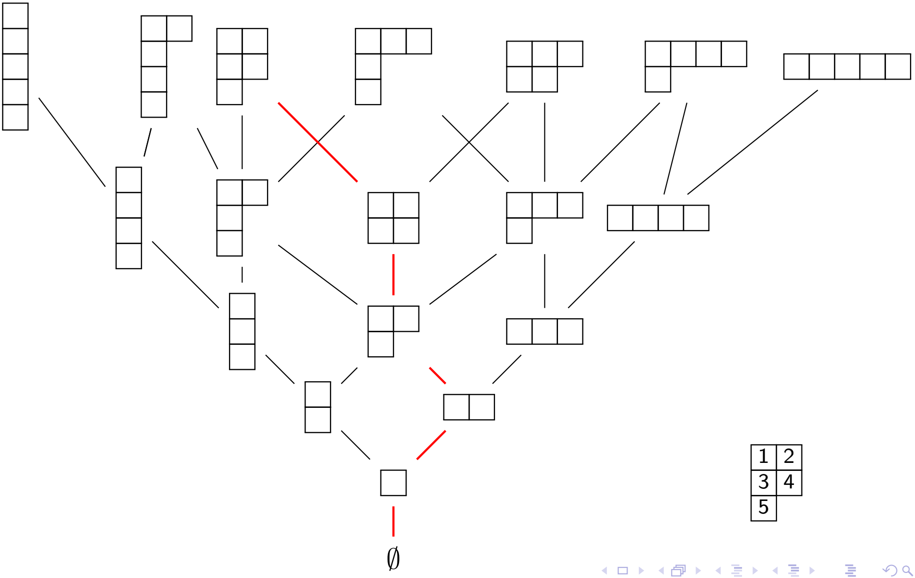
2	5
4	

1	4
2	5
3	

- Thus $f^{(2,2,1)} = 5$

An example of U^5



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Hook Length Formula

- Subgoal: How many SYT are there for a single shape λ?
- Let u be a square of the Young diagram of the partition λ. Define the hook $H(u)$ of u to be the set of all squares directly to the right of u or directly below u, including u itself and $h(u) = |H(u)|$.

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- Below is the diagram of the partition $(4, 2, 2)$.

6	5	2	1
3	2		
2	1		

Theorem: Hook Length Formula

- **Theorem:** Let $\lambda \vdash n$. Then

$$f^\lambda = \frac{n!}{\prod_{u \in \lambda} h(u)}.$$

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- Hence, there are 56 SYT of shape $(4, 2, 2)$

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1	

$$f^{(2,2,1)} = \frac{(2+2+1)!}{4 \cdot 3 \cdot 2 \cdot 1 \cdot 1} = 5.$$

- Hence, there are 5 SYT of shape $(2, 2, 1)$

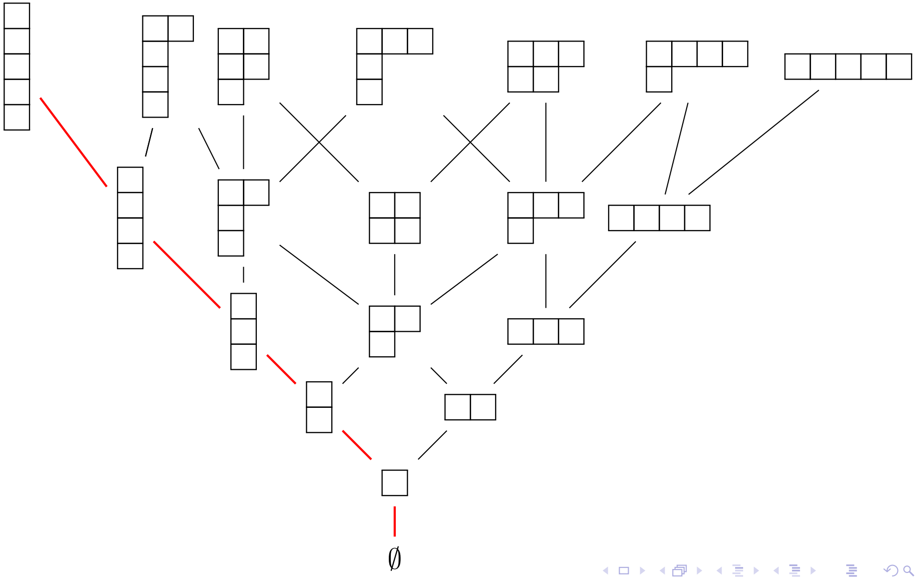
Goal

- Goal: Count the number of walks of type U^n. Call this number T_n.
- Let's do the case where $n = 5$.

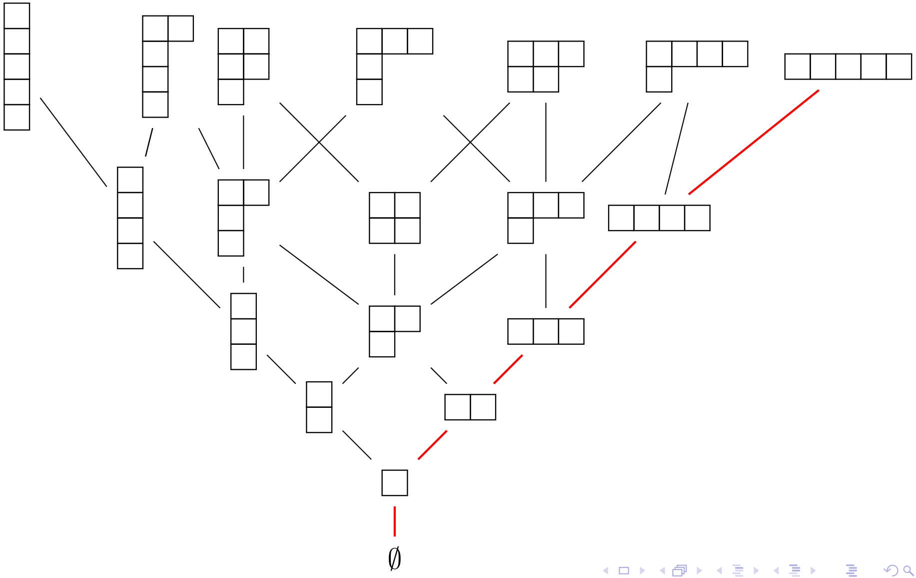
$$T_n = \sum_{\lambda \vdash 5} f^\lambda = \sum_{\lambda \vdash 5} \frac{n!}{\prod_{u \in \lambda} h(u)}$$

Now we calculate T_5

$\lambda = (11111)$



$\lambda = (5)$



U^5

$$\begin{array}{|c|c|}\hline 5 & 1 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1} 2 & \multicolumn{1}{c}{} \\ \cline{1-1} 1 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}$$

$$f^{(2,1,1,1)} = \frac{5!}{5 \cdot 3 \cdot 2 \cdot 1 \cdot 1} = 4.$$

$$\begin{array}{|c|c|c|}\hline 5 & 2 & 1 \\ \hline 2 & \multicolumn{2}{c}{} \\ \cline{1-1} 1 & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array}$$

$$f^{(3,1,1)} = \frac{5!}{5 \cdot 2 \cdot 2 \cdot 1 \cdot 1} = 3.$$

$$\begin{array}{|c|c|c|}\hline 4 & 3 & 1 \\ \hline 2 & 1 & \multicolumn{1}{c}{} \\ \cline{1-2} \end{array}$$

$$f^{(3,1,1)} = \frac{5!}{4 \cdot 3 \cdot 2 \cdot 1 \cdot 1} = 5.$$

$$\begin{array}{|c|c|c|c|}\hline 5 & 3 & 2 & 1 \\ \hline 1 & \multicolumn{3}{c}{} \\ \cline{1-1} \end{array}$$

$$f^{(4,1)} = \frac{5!}{5 \cdot 3 \cdot 2 \cdot 1 \cdot 1} = 4.$$

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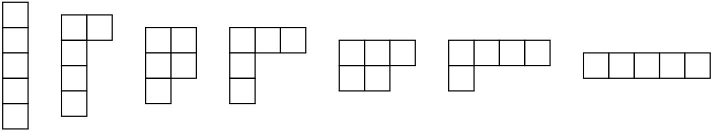

$$T_5 = 1 + 4 + 5 + 3 + 5 + 4 + 1 = 23.$$

Thank you

Citation:
Stanley, Richard, "Topics in Algebraic Combinatorics"