The Interplay Between Promotion and Rowmotion

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Abstract

This paper explores the relationship between promotion and rowmotion within the framework of poset theory. Theorem 2.12, which is about decomposition of linear extensions of a poset and is proved by promotion, is originally from the paper *A Recurrence for Linear Extensions* by Paul Edelman, Takayuki Hibi, and Richard P. Stanley. Most of the definitions and theorems are from the paper *Promotion and Rowmotion* by Jessica Striker and Nathan Williams. We examine how promotion, defined as a specific alteration of linear extensions in a poset, intertwines with rowmotion, a map translating order ideals in a poset. Eventually, it leads to the main result of this paper: Theorem 3.3, which shows that promotion and rowmotion are conjugate elements in the toggle group of an rc-poset.

1 Introduction

A *poset* is a set $\mathcal{P}$ equipped with a binary relation $\leq$ that is reflexive, antisymmetric, and transitive. M.-P. Schützenberger defined linear extensions of a poset with n elements as the bijections of the poset with the set of integers $\{1, 2, \ldots, n\}$ that preserve the order.

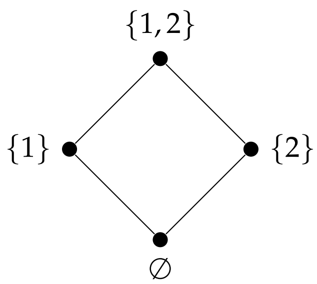
Definition 1.1 (Linear Extension)**.** Let $\mathcal{P}$ be a poset with n elements. A linear extension is a bijection $\mathcal{L} : \mathcal{P} \to [n]$ such that if $x < y$ in $\mathcal{P}$, then $\mathcal{L}(x) < \mathcal{L}(y)$.

2 Promotion and Rowmotion

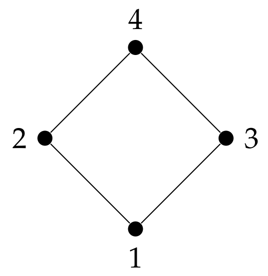
Definition 2.1 (Promotion)**.** Let $\mathcal{L}$ be a linear extension of a poset $\mathcal{P}$ and let ρ_i act on $\mathcal{L}$ by switching i and $i+1$ if they are not the labels of two elements with a covering relation. We define the *promotion* of $\mathcal{L}$ to be

$$\mathrm{Pro}(\mathcal{L}) = \rho_{n-1}\rho_{n-2}\cdots\rho_1(\mathcal{L}).$$

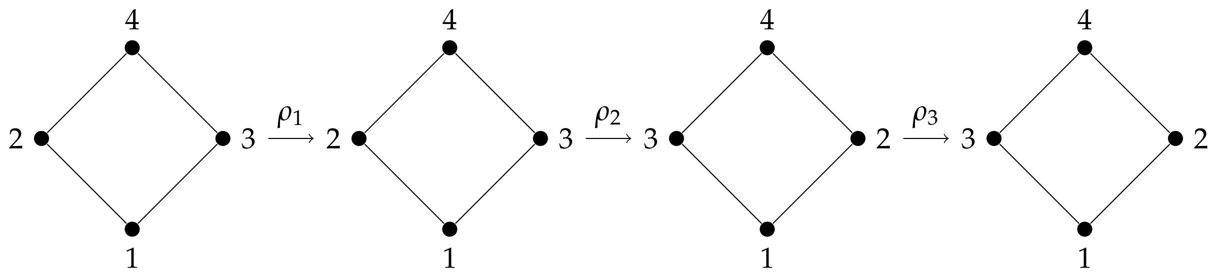
Example 2.2. Here we will use a Hasse diagram of B_2 to illustrate the promotion of a linear extension. The Graph below is the hass diagram of poset of B_2.



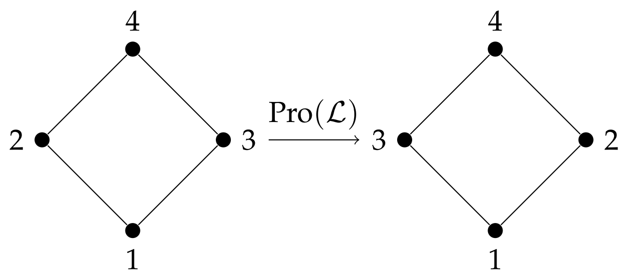
We define a linear extension of B_2 as $\mathcal{L} : \{\emptyset, \{1\}, \{2\}, \{1,2\}\} \to [4]$ such as $\mathcal{L}(\emptyset) = 1, \mathcal{L}(\{1\}) = 2, \mathcal{L}(\{2\}) = 3, \mathcal{L}(\{1,2\}) = 4$. Then we have



Here is the each ρ_i acts on $\mathcal{L}$:



Thus, we have



Through the above example, we can see that the promotion of a linear extension is a linear extension.

Proposition 2.3. For a finite poset, the promotion of a linear extension is a linear extension.

Proof. (to be added) We will prove it with mathematical contradiction. For a finite poset $\mathcal{P}$, we define a linear extension of $\mathcal{P}$ as $\mathcal{L} : \mathcal{P} \to [n]$ such as $\mathcal{L}(x) < \mathcal{L}(y)$ in $\mathcal{L}(\mathcal{P})$, if $x < y$ in $\mathcal{P}$. And we define the promotion of $\mathcal{L}$ as $\text{Pro}(\mathcal{L})$. Suppose that $\text{Pro}(\mathcal{L})$ is not a linear extension of $\mathcal{P}$. Then, there exists $x, y \in \mathcal{P}$ such that $\text{Pro}(\mathcal{L})(x) > \text{Pro}(\mathcal{L})(y)$ but $x < y$ in $\mathcal{P}$. Since $x < y$ in $\mathcal{P}$, we have $\mathcal{L}(x) < \mathcal{L}(y)$ in $\mathcal{L}(\mathcal{P})$. If the positions of $\mathcal{L}(x)$ and $\mathcal{L}(y)$ are not changed under the promotion, then we have $\text{Pro}(\mathcal{L})(x) = \mathcal{L}(x) < \mathcal{L}(y) = \text{Pro}(\mathcal{L})(y)$, which is a contradiction. Hence, at least one of the positions of $\mathcal{L}(x)$ and $\mathcal{L}(y)$ is changed under the promotion. If the position of $\mathcal{L}(x)$ is changed in $\text{Pro}(\mathcal{L})$, then, according to the definition of promotion, it can only switch position with $\mathcal{L}(x) - 1 \in [n]$ or $\mathcal{L}(x) + 1 \in [n]$. Similarly, $\mathcal{L}(y)$ can only switch position with $\mathcal{L}(y) - 1 \in [n]$ or $\mathcal{L}(y) + 1 \in [n]$. If there is only one of $\mathcal{L}(x)$ or $\mathcal{L}(y)$, we can see that $\text{Pro}(\mathcal{L})(x) < \text{Pro}(\mathcal{L})(y)$. Hence, both of $\mathcal{L}(x)$ and $\mathcal{L}(y)$ are changed in $\text{Pro}(\mathcal{L})$ such as $\mathcal{L}(x)$ switch with $\mathcal{L}(x) + 1$ and $\mathcal{L}(y)$ switch with $\mathcal{L}(y) - 1$. Hence, we have $\text{Pro}(\mathcal{L}(y)) = \mathcal{L}(y) - 1 < \text{Pro}(\mathcal{L}(X)) = \mathcal{L}(x) + 1$, which indicates that x is covered by y in $\mathcal{P}$, which is a contradiction. $\square$

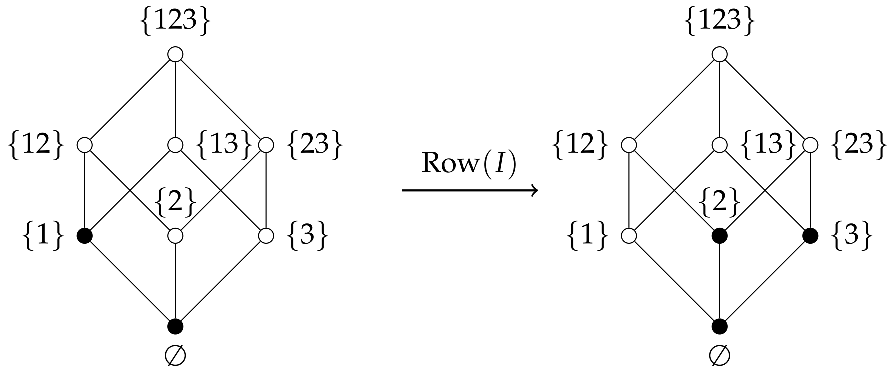
Definition 2.4 (Order Ideal)**.** An *order ideal* of a poset $\mathcal{P}$ is a set $I \subseteq \mathcal{P}$ such that if $p \in I$ and $p' \leq p$, then $p' \in I$. We write $J(\mathcal{P})$ for the set of all order ideals of $\mathcal{P}$.

Example 2.5. In B_3, $\{\varnothing, \{1\}, \{2\}, \{1,2\}\}$ is an order ideal.

After we define order ideal, we can introduce the concept of **Rowmotion**. Since I like the way Professor Tom Roby defined Rowmotion, I will use his definition in one of his presentation slides. In his original slide, he used the word 'classical rowmotion' to distinguish from other types of rowmotions; however, I will simply use the word 'rowmotion'.

Definition 2.6 (Rowmotion)**.** Let $\mathcal{P}$ be a finite poset. **Rowmotion** is the map $\mathrm{Row}\colon J(\mathcal{P}) \to J(\mathcal{P})$ which sends *every order ideal* I to the order ideal obtained as follows: Let M be the set of minimal elements of the complement $\mathcal{P} \setminus I$. Then, $\mathrm{Row}(I)$ shall be the order ideal generated by these elements (i.e., the set of all $w \in \mathcal{P}$ such that there exists an $m \in M$ such that $w \leq m$).

Example 2.7. Here is an example of Rowmotion in context of the poset is B_3. Let order ideal I to be $\{\emptyset, \{1\}\}$. Firstly, we can see that the complement of I is $\{\{2\}, \{3\}, \{1,3\}, \{1,2\}, \{2,3\}, \{1,2,3\}\}$. The minimal elements of the complement are $\{2\}$ and $\{3\}$. The order generated by $\{2\}$ and $\{3\}$ is $\{\emptyset, \{2\}, \{3\}\}$, which are filled in black in the graph on the right hand side.



Definition 2.8 (Toggle (Cameron and Fon-der-Flaass))**.** For each $p \in \mathcal{P}$, define $t_p\colon J(\mathcal{P}) \to J(\mathcal{P})$ to act by toggling p if possible. That is, if $I \in J(\mathcal{P})$,

$$t_p(I) = \begin{cases} I \cup \{p\} & \text{if } p \notin I \text{ and if } p' < p \text{ then } p' \in I, \\ I \setminus \{p\} & \text{if } p \in I \text{ and if } p' > p \text{ then } p' \notin I, \\ I & \text{otherwise.} \end{cases}$$

Definition 2.9 (Toggle group (Cameron and Fon-der-Flaass))**.** The toggle group $T(\mathcal{P})$ of a poset $\mathcal{P}$ is the subgroup of the permutation group $\mathfrak{S}_{J(\mathcal{P})}$ generated by $\{t_p\}_{p\in\mathcal{P}}$. Note that $T(\mathcal{P})$ has the following obvious relations (which do not constitute a full presentation).

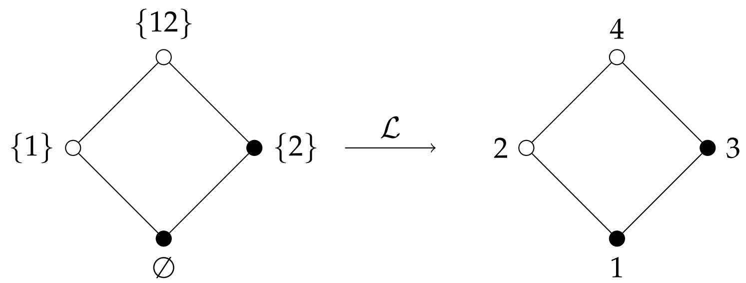
1. $t_p^2 = 1$, and
2. $(t_p t_{p'})^2 = 1$ if p and p' do not have a covering relation.

Theorem 2.10 (Cameron and Fon-der-Flaass)**.** *Fix a linear extension $\mathcal{L}$ of a poset $\mathcal{P}$ with n elements. Let $I \in J(\mathcal{P})$. Then*

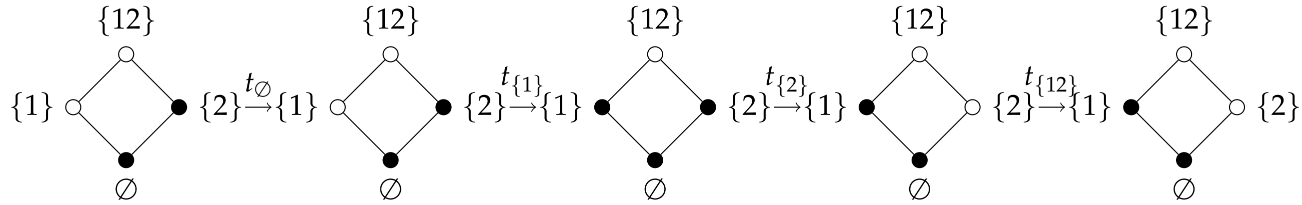
$$t_{\mathcal{L}^{-1}(1)} t_{\mathcal{L}^{-1}(2)} \cdots t_{\mathcal{L}^{-1}(n)}(I)$$

acts as Rowmotion.

Example 2.11. To have illustrate the idea, we will use B_2 again with order ideal $I = \{\emptyset, \{2\}\}$ as an easy example. Firstly, we can have the complement of I is $\{\{1\}, \{1,2\}\}$. The minimal element of the complement is $\{1\}$, and the order ideal generated by $\{1\}$ is $\{\emptyset, \{1\}\}$(i.e, $\mathrm{Row}(I) = \{\emptyset, \{1\}\}$). Let linear extension $\mathcal{L}$ defined as below:



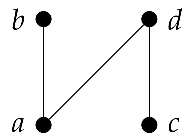
We can observe that $\mathcal{L}^{-1}(1) = \emptyset, \mathcal{L}^{-1}(2) = \{1\}, \mathcal{L}^{-1}(3) = \{2\}, \mathcal{L}^{-1}(4) = \{12\}$.



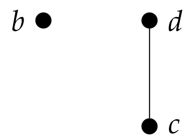
Theorem 2.12 (Edelman, Hibi, Stanley)**.** Let $\mathcal{P}$ be a finite poset and denote $\mathcal{L}(\mathcal{P})$ as the set of all linear extensions of $\mathcal{P}$. We let A to be the set of all minimal elements of $\mathcal{P}$. Then,

$$|\mathcal{L}(\mathcal{P})| = \sum_{x \in A} |\mathcal{L}(\mathcal{P} \setminus \{x\})|.$$

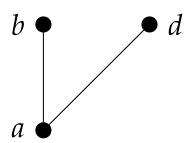
Example 2.13. Boolean algebra B_n is a special case here, since the only minimal element of B_n is the minimum element $\emptyset$. Through observation, we can see that when we remove $\emptyset$ from B_n and create a new poset B_n', the number of linear extensions of B_n' equals to the number of linear extensions of B_n, which satisfies the theorem. In general, this theorem will not give us a lot of information if the poset has the minimum element. The below example is a poset $\mathcal{P}$ that has more than one minimal element.



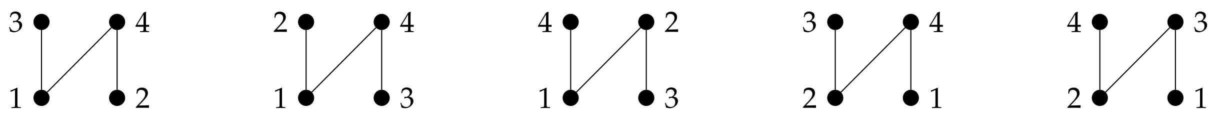
We can see that the minimal element set A of $\mathcal{P}$ is $\{a, c\}$. When we remove one of the minimal elements, say a, from $\mathcal{P}$, we have a new poset $\mathcal{P}_1$ as below:



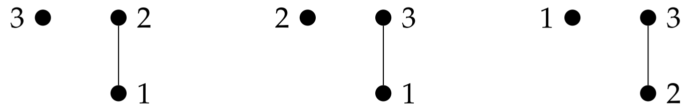
When we remove c from $\mathcal{P}$, we have a new poset $\mathcal{P}_2$ as below:



Now we list all the linear extensions of $\mathcal{P}$ below and there are five of them.



For linear extensions of $\mathcal{P}_1$, we have the following:



For linear extensions of $\mathcal{P}_2$, we have the following:

We have 5 add the number of linear extensions of $\mathcal{P}_1$ and $\mathcal{P}_2$ together, which is the number of linear extensions of $\mathcal{P}$.

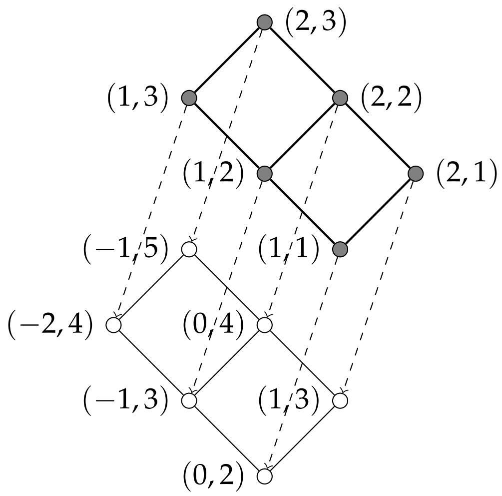
To prove our main result, we need to define a new poset called rowed-and-columned poset.

Definition 2.14. Let $\Pi \subseteq \mathbb{R}^2$ be the set of points in the integer span of $(2,0)$ and $(1,1)$. A rowed-and-columned (rc) poset $\mathcal{R}$ is a finite poset together with a map $\pi : \mathcal{R} \to \Pi$, where if $p_1, p_2 \in \mathcal{R}$, p_1 covers p_2, and $\pi(p_1) = (i,j)$, then $\pi(p_2) = (i+1, j-1)$ or $\pi(p_2) = (i-1, j-1)$. For $p \in \mathcal{R}$, we call $\pi(p) \in \Pi$ the *position* of p.

Definition 2.15. Give an rc-poset $\mathcal{R}$, the jth *row* of$\mathcal{R}$ is the set of elements of $\mathcal{R}$ in positions $\{(i,j)\}_i$. The ith *column* of an rc-poset is the set of elements of $\mathcal{R}$ in positions $\{(i,j)\}_j$.

Note 2.16. If we strictly follow the definition above, we will encounter the situation that we need to use negative index to represent the row and column, like the example above. Hence, we will loose the definition here. We will call the row at the bottom as the first row and the row at the top as the last row. Similarly, we will call the column at the left as the first column and the column at the right as the last column.

Example 2.17. Product of two chains $[n] \times [k]$ with map $\pi(i,j) = (i-j, i+j)$ is an rc-poset. Here is an example of $[2] \times [3]$ with $\pi(i,j) = (i-j, i+j)$.



The lattice on the top of the example above is the product of two chains $[2] \times [3]$, and their cover relations are denoted by the solid lines. The map π is denoted by the dashed lines. The lattice at the bottom are the positions of the elements in the product of two chains $[2] \times [3]$. But keep in mind that the bottom lattice is not the element of rc-posets, but they can tell us whether some elements on the top lattices are in the same row or column.
To identify each row of this rc-poset, we can look at the index of the elements in the bottom lattice. Since $(1,1)$ on the top is the only element with index $(i,2)$ at the bottom, we can say that $(1,1)$ is the only element in the first row. For the second row, we have $(1,2)$ and $(2,1)$ on the top, and they share the positions $(i,3)$ at the bottom. For the third row, we have $(1,3)$, $(2,2)$, since they share the positions $(i,4)$ at the bottom. For the fourth row, we only have $(2,3)$.
To identify the columns, we start from the columns from the left. The first column is $\{(1,3)\}$, since it is the only element with index $(-2,i)$ at the bottom. The second column is $\{(1,2),(2,3)\}$, since they are the only elements with index $(-1,i)$ at the bottom. The third column is $\{(1,1),(2,2)\}$, since they are the only elements with index $(0,i)$ at the bottom. The last column is $\{(2,1)\}$, since it is the only element with index $(1,i)$ at the bottom.

Definition 2.18. Given $\mathcal{R}$ is an rc-poset and $I \in J(\mathcal{R})$, let $r_i(I) = \prod t_p(I)$, where the product is over all elements in ith row, and let $c_i(I) = \prod t_p(I)$, where the product is over all elements in ith column.

Proposition 2.19. Given the definition above, we have the following properties:

1. $r_i^2 = c_i^2 = 1$, and
2. if $|i-j| > 1$, $(r_i r_j)^2 = (c_i c_j)^2 = 1$.

Proof. If two elements are in the same row, then they are not comparable. According to Definition 2.9, if two elements have no covering relation, then their toggle action commute. Then we have $r_i^2 = 1$ since r_i is the product of all toggles on the elements in the same row and all toggles are commuting. Similarly, we have $c_i^2 = 1$ since all elements in the same column do not have covering relations either. It will be true for the second property as well, if two elements are not from the same row or column, and the rows or columns are not adjacent, then they do not share covering relations and their toggle actions commute. Hence, we have $(r_i r_j)^2 = 1$ and $(c_i c_j)^2 = 1$. □

we will borrow the idea from Theorem 2.10 and define the promotion and rowmotion as the element of the toggle group of an rc-poset with n rows and k columns.

Definition 2.20. Given an rc-poset $\mathcal{R}$ with n rows and k columns, and $\nu \in \mathfrak{S}_k$ and $\omega \in \mathfrak{S}_n$, we define the action Pro_ν on $I \in J(\mathcal{R})$ to be

$$\mathrm{Pro}_\nu(I) = c_{\nu(k)} \cdots c_{\nu(1)}(I),$$

and define the action Pro_ω on $I \in J(\mathcal{R})$ to be

$$\mathrm{Row}_\omega(I) = \prod_{i=1}^{n} r_{\omega(i)}(I).$$

Note 2.21. The way we define Pro_ν here is not strictly following the idea of group operation. Instead, we treat it like a composition of functions to match the definition of promotion we defined in Definition 2.1. However, the promotion we defined at the beginning of this section is not an action on order ideals; instead, it acts on linear extensions. Here, since the idea of toggling involved in Pro_ν, it is reasonable to define it as an action on order ideals.

Corollary 2.22. On an rc-poset $\mathcal{R}$ with n rows, we let $I \in J(\mathcal{P})$, $Row_e(I)$ acts as rowmotion where $e \in \mathfrak{S}_n$ is the identity permutation.

Proof. Since $\mathcal{R}$ has n rows, we have $\mathrm{Row}_e(I) = r_{e(1)}(I) \cdots r_{e(n)}(I) = r_1(I) \cdots r_n(I)$. Here r_1 is the toggle action on elements on the lowest row, and r_n is the toggle action on the elements on the highest row. We already know that any two elements in the ith row are not comparable, so we have $t_p(I)t_{p'}(I) = t_{p'}(I)t_p(I)$ for any p, p' on the ith row. Thus, we have

$$\mathrm{Row}_e(I) = r_1(I) \cdots r_n(I) = t_{p_{11}}(I) \cdots t_{p_{1n_1}}(I) \cdots t_{p_{n1}}(I) \cdots t_{p_{nn_n}}(I).$$

Suppose that $\mathcal{R}$ has m elements, we define a map $\mathcal{L} : \mathcal{R} \to [m]$ following the order below:

$$\mathcal{L}(p_{11}) = 1 < \cdots < \mathcal{L}(p_{1n_1}) < \cdots < \mathcal{L}(p_{n1}) < \cdots < \mathcal{L}(p_{nn_n}) = m$$

Through observation, we can see that the map $\mathcal{L}$ is a linear extension. According to Theorem 2.10, we can see that $Row_e(I)$ is the same as the rowmotion on I. □

Since we did not define an order ideal under promotion, we need to define it here.

Definition 2.23. Given an rc-poset $\mathcal{R}$ and an order ideal $I \subseteq J(\mathcal{R})$, define the promotion of I to be $\mathrm{Pro}(I) = \mathrm{Pro}_e(I)$, where e is the identity permutation.

3 Main result

We need to use the following Lemma 3.1 and Lemma 3.2 to prove the main result, which is Theorem 3.3.

Lemma 3.1. Let G be a group acting on a set X, and let g_1 and $g_2 = gg_1g^{-1}$ be conjugate elements in G. Then $x \mapsto gx$ gives an equivariant bijection between X under g_1 and X under g_2.

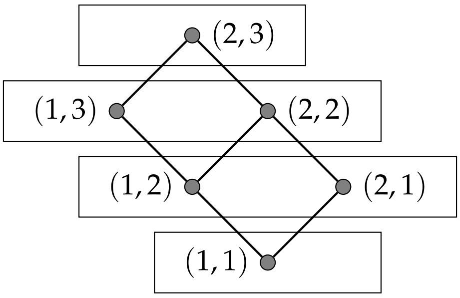
We can describe this lemma by the following commutative diagram:

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "g_1"'] & X \arrow[d, "g_2 = gg_1g^{-1}"] \\ X \arrow[r, "g"'] & X \end{tikzcd}$$

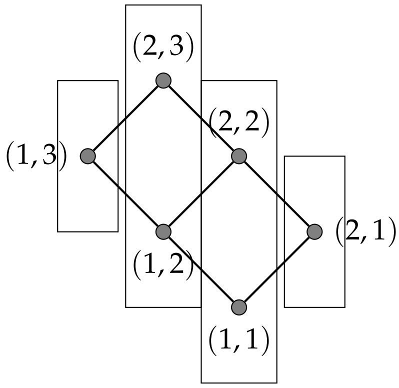
Lemma 3.2 (Striker, Williams)**.** Let G be a group whose generators $g_1, \ldots, g_n$ satisfy $g_i^2 = 1$ and $(g_ig_j)^2 = 1$ if $|i - j| > 1$. Then for any $\omega, \nu \in \mathfrak{S}_n$, $\prod_i g_{\omega(i)}$ and $\prod_i g_{\nu(i)}$ are conjugate.

Theorem 3.3 (Striker, Williams)**.** *F or any rc-poset $\mathcal{R}$ and any $\omega \in \mathfrak{S}_n$ and $\nu \in \mathfrak{S}_k$, there is an equivariant bijection between $J(\mathcal{R})$ under Pro_ω and $J(\mathcal{R})$ under Row_ω.*

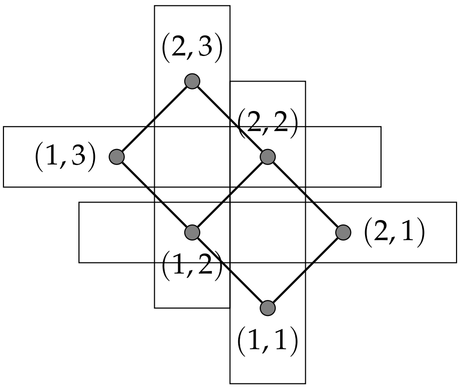
Proof. (Sketch) Recall Definition 2.18, we have $r_i^2 = c_i^2 = 1$, and if $|i - j| > 1$, $(r_ir_j)^2 = (c_ic_j)^2 = 1$. According to Lemma 3.2, we have $\langle r_{\omega(i)} \rangle$ and $\langle r_{\nu(i)} \rangle$ are conjugate, and $c_{\omega(i)}$ and $c_{\nu(i)}$ are conjugate. Since $\mathrm{Row}_\omega(I)$ is generated by $\{r_\omega(I)\}$ and $\mathrm{Row}_\nu(I)$ is generated by $\{r_\nu(I)\}$, according to Lemma 3.1, we have an equivalent bijection between $J(\mathcal{R})$ under Pro_ω and $J(\mathcal{R})$ under Pro_ν. For the rest of sketch of the proof, it will be better use example to illustrate the idea. We will use the rc-poset $\mathcal{R} = [2] \times [3]$ with $\pi(i, j) = (i - j, i + j)$ as an example. Since we already know that the positions of elements in $\mathcal{R}$, we firstly circle all the elements in the same row in rectangles. One thing is all elements in the same row are not comparable, so all the toggles in the same row commute.



Then we circle all the elements in the same column in rectangles. Similarly, all the elements in the same column do not share the covering relations, so all the toggles in the same column commute as well.



Now we make an intersection of the rectangles we have drawn above. Now you can see that the toggles commute for all the elements at the second row and second column. And toggles commute for all the elements at the third row and third column.



In general, t_p for all p in an odd column or row commute with one another. Similarly, t_p for all p in an even column or row commute with one another. Thus, for any $\nu \in \mathfrak{S}_k$ and $\omega \in \mathfrak{S}_n$, we have $\mathrm{Pro}_\nu(I)$ is equivalent to $\mathrm{Row}_\omega(I)$ by commuting the toggles in the odd rows or columns and commuting the toggles in the even rows or columns. $\square$

4 Summary

We investigates the connections between promotion and rowmotion. It begins by defining key concepts: promotion, order ideals, rowmotion, the toggle group of a poset, and a special poset called rowed-and-columned poset (rc-poset). The paper references the work of Edelman, Hibi, and Stanley on linear extensions, and Striker and Williams on promotion and rowmotion. The main result, Theorem 3.3, establishes that promotion and rowmotion are conjugate in the toggle group of an rc-poset. This is demonstrated through examples, including the rc-poset $[2] \times [3]$. The proof strategy involves group action, conjugacy, and toggle group of rc-posets.

References

[1] Jessica Striker, Nathan Williams, *Promotion and rowmotion*, European Journal of Combinatorics, Elsevier, Volume 58, 2016, Pages 238–266.

[2] Paul Edelman, Takayuki Hibi, and Richard P. Stanley, *A Recurrence for Linear Extensions*, Order, Volume 6, 1989, Pages 15–18, Kluwer Academic Publishers.